

# MULTIVARIABLE AND VECTOR ANALYSIS

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## Chapter 6

### CHANGE OF VARIABLES

#### 6.1. Introduction

We consider double integrals of the type

$$\iint_S f(x, y) \, dx \, dy,$$

where  $f : S \rightarrow \mathbb{R}$  is continuous in the region  $S \subseteq \mathbb{R}^2$ . Suppose that we wish to make a substitution  $x = x(u, v)$  and  $y = y(u, v)$ . Then as in the theory of real valued functions of one real variable, we expect to have

$$(1) \quad \iint_S f(x, y) \, dx \, dy = \iint_{S^*} f(x(u, v), y(u, v)) J(u, v) \, du \, dv,$$

where  $J(u, v)$  is a term arising from the substitution, and where  $S^*$  is a region in the  $uv$ -plane corresponding to the region  $S$  in the  $xy$ -plane.

Before we go any further, let us consider the following simple example in the theory of real valued functions of one real variable.

EXAMPLE 6.1.1. To evaluate the integral

$$\int_0^1 \sqrt{1-x^2} \, dx,$$

we may use the substitution  $x = \cos \theta$ . Then  $dx = -\sin \theta \, d\theta$ , and  $\theta = \pi/2$  when  $x = 0$  and  $\theta = 0$  when  $x = 1$ . Hence

$$\int_0^1 \sqrt{1-x^2} \, dx = \int_{\pi/2}^0 \sqrt{1-\cos^2 \theta} (-\sin \theta) \, d\theta = \int_0^{\pi/2} \sin^2 \theta \, d\theta = \dots = \frac{\pi}{4}.$$

Clearly there is great potential for making a mistake by omitting a minus sign somewhere in the argument. We may instead proceed in the following way. Consider the function  $T : [0, \pi/2] \rightarrow [0, 1]$ , given by  $T(\theta) = \cos \theta$  for every  $\theta \in [0, \pi/2]$ . Clearly this function is one-to-one and onto. We may well suspect that for any continuous function  $f : [0, 1] \rightarrow \mathbb{R}$ , we have

$$\int_{[0,1]} f(x) \, dx = \int_{[0,\pi/2]} f(T(\theta)) \left| \frac{dT}{d\theta} \right| d\theta.$$

Indeed, this is true when  $f(x) = \sqrt{1-x^2}$  for every  $x \in [0, 1]$ , and is in fact true for every continuous function  $f : [0, 1] \rightarrow \mathbb{R}$ .

## 6.2. Planar Transformations

In this section, we shall study functions of the type  $T : S^* \rightarrow S$ , where  $S^*$  and  $S$  are sets in  $\mathbb{R}^2$ .

Suppose that  $S^* \subseteq \mathbb{R}^2$  is given. Consider a function  $T : S^* \rightarrow \mathbb{R}^2$ . Then for any  $(u, v) \in S^*$ , the point  $T(u, v)$  is called the image of  $(u, v)$  under  $T$ , and the set

$$T(S^*) = \{T(u, v) : (u, v) \in S^*\}$$

is called the range of the function  $T$ .

EXAMPLE 6.2.1. Suppose that  $S^* = [0, 1] \times [0, 2\pi]$ . The function  $T : S^* \rightarrow \mathbb{R}^2 : (r, \theta) \mapsto (r \cos \theta, r \sin \theta)$  has range

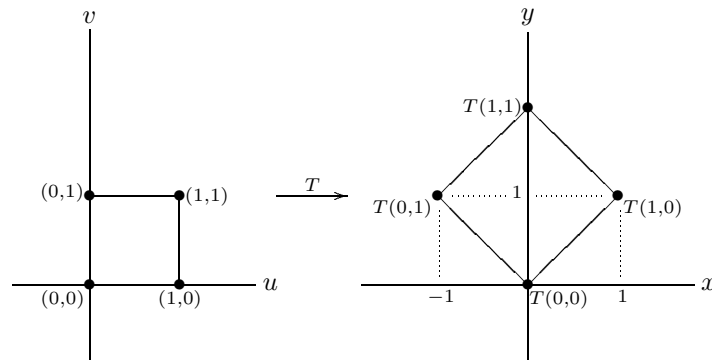
$$T(S^*) = \{(r \cos \theta, r \sin \theta) : (r, \theta) \in [0, 1] \times [0, 2\pi]\},$$

the closed disc in  $\mathbb{R}^2$  with radius 1 and centre  $(0, 0)$ .

EXAMPLE 6.2.2. Suppose that  $S^* = [0, 1] \times [0, 1]$ . Consider the function  $T : S^* \rightarrow \mathbb{R}^2$ , given for every  $(u, v) \in S^*$  by  $T(u, v) = (x, y)$ , where

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \quad \text{and} \quad x + iy = (1 + i)(u + iv).$$

Clearly  $T(S^*)$  is a square with vertices  $(0, 0)$ ,  $(0, 2)$  and  $(\pm 1, 1)$ .



DEFINITION. A function  $T : S^* \rightarrow \mathbb{R}^2$  is said to be one-to-one if for every  $(u', v'), (u'', v'') \in S^*$ , we have  $(u', v') = (u'', v'')$  whenever  $T(u', v') = T(u'', v'')$ . In other words, no two distinct  $(u', v'), (u'', v'') \in S^*$  can have the same image.

EXAMPLE 6.2.3. In Example 6.2.1, the function  $T : S^* \rightarrow \mathbb{R}^2$  is not one-to-one. It is easy to check that  $T(0, 0) = T(0, \pi) = 0$ .

EXAMPLE 6.2.4. In Example 6.2.2, it is easily seen that the function  $T : S^* \rightarrow \mathbb{R}^2$  is a rotation about the point  $(0, 0)$  through an angle  $\pi/4$ , together with a magnification by  $\sqrt{2}$  about the point  $(0, 0)$ . It is clear that if  $T(u', v') = (x', y')$ ,  $T(u'', v'') = (x'', y'')$  and  $(x', y') = (x'', y'')$ , then since the matrix  $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$  is invertible, we must have  $(u', v') = (u'', v'')$ . Hence  $T : S^* \rightarrow \mathbb{R}^2$  is one-to-one.

EXAMPLE 6.2.5. Let us vary Example 6.2.1, and consider the rectangle  $S^* = (0, 1] \times [0, 2\pi)$ . Then the function  $T : S^* \rightarrow \mathbb{R}^2 : (r, \theta) \mapsto (r \cos \theta, r \sin \theta)$  is one-to-one, with range

$$T(S^*) = \{(r \cos \theta, r \sin \theta) : (r, \theta) \in (0, 1] \times [0, 2\pi)\},$$

the closed disc in  $\mathbb{R}^2$  with radius 1 and centre  $(0, 0)$ , punctured at the point  $(0, 0)$ .

In practice, our problem is the following: Given a region  $S \subseteq \mathbb{R}^2$ , we need to find a region  $S^* \subseteq \mathbb{R}^2$  and a one-to-one function  $T : S^* \rightarrow \mathbb{R}^2$  such that  $T(S^*) = S$ . It is therefore convenient to consider functions of the type  $T : S^* \rightarrow S$  instead of  $T : S^* \rightarrow \mathbb{R}^2$ . Note that this change does not affect our definition of one-to-one functions given earlier, and we can rephrase it as follows.

DEFINITION. A function  $T : S^* \rightarrow S$  is said to be one-to-one if for every  $(u', v'), (u'', v'') \in S^*$ , we have  $(u', v') = (u'', v'')$  whenever  $T(u', v') = T(u'', v'')$ .

To characterize the property  $T(S^*) = S$ , we have the following definition.

DEFINITION. A function  $T : S^* \rightarrow S$  is said to be onto if for every  $(x, y) \in S$ , there exists  $(u, v) \in S^*$  such that  $T(u, v) = (x, y)$ .

EXAMPLE 6.2.6. Suppose that  $S^* = [0, 1] \times [0, 2\pi]$  and  $S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ . Then the function  $T : S^* \rightarrow \mathbb{R}^2 : (r, \theta) \mapsto (r \cos \theta, r \sin \theta)$  is onto.

EXAMPLE 6.2.7. In Example 6.2.2, if we change the codomain of the function  $T$  to the closed rectangle  $S$  with vertices  $(0, 0)$ ,  $(0, 2)$  and  $(\pm 1, 1)$ , then  $T : S^* \rightarrow S$  is one-to-one and onto.

EXAMPLE 6.2.8. Suppose that  $S^* = (0, 1] \times [0, 2\pi)$  and  $S = \{(x, y) \in \mathbb{R}^2 : 0 < x^2 + y^2 \leq 1\}$ . Then the function  $T : S^* \rightarrow \mathbb{R}^2 : (r, \theta) \mapsto (r \cos \theta, r \sin \theta)$  is one-to-one and onto.

### 6.3. The Jacobian

In this section, we shall turn our attention to the term  $J(u, v)$  in the expression (1). We shall begin by studying the special case when  $f(x, y) = 1$  for every  $(x, y) \in S$ , so that the expression (1) is of the form

$$(2) \quad \iint_S dx dy = \iint_{S^*} J(u, v) du dv.$$

Note that the left hand side of (2) represents the area of  $S$ .

EXAMPLE 6.3.1. We study rectangular coordinates  $(x, y)$  and polar coordinates  $(r, \theta)$ , related by

$$x = r \cos \theta \quad \text{and} \quad y = r \sin \theta.$$

Consider the unit disc

$$S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}.$$

Let

$$S^* = \{(r, \theta) \in \mathbb{R}^2 : 0 \leq r \leq 1 \text{ and } 0 \leq \theta \leq 2\pi\}.$$

Then the function

$$T : S^* \rightarrow \mathbb{R}^2 : (r, \theta) \mapsto (r \cos \theta, r \sin \theta)$$

is one-to-one and onto, provided that we remove the lines  $r = 0$  and  $\theta = 2\pi$  from  $S^*$  and the point  $(x, y) = (0, 0)$  from  $S$ . Clearly

$$\iint_S dx dy = \pi.$$

On the other hand,

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \det \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \det \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix} = r.$$

By Fubini's theorem, we have

$$\iint_{S^*} \left| \frac{\partial(x, y)}{\partial(r, \theta)} \right| dr d\theta = \iint_{S^*} r dr d\theta = \int_0^{2\pi} \left( \int_0^1 r dr \right) d\theta = \pi.$$

Here we detect a small problem. The function  $T : S^* \rightarrow S$  may not be one-to-one and onto, unless we remove some subsets of  $S^*$  and  $S$ . Let us first of all consider such exceptional subsets.

**DEFINITION.** Suppose that  $S \subseteq \mathbb{R}^2$ . Then we say that a subset  $E \subseteq S$  is exceptional if  $E$  is contained in a finite union of curves of type 1 or 2 in  $S$ .

**DEFINITION.** Suppose that  $S^*$  and  $S$  are sets in  $\mathbb{R}^2$ . Then we say that a function  $T : S^* \rightarrow S$  is essentially one-to-one and onto if there exist exceptional subsets  $E^* \subseteq S^*$  and  $E \subseteq S$  such that the function  $T : S^* \setminus E^* \rightarrow S \setminus E$  is one-to-one and onto.

**EXAMPLE 6.3.2.** In Example 6.3.1, we can take

$$E^* = \{(r, \theta) \in S^* : r = 0 \text{ or } \theta = 2\pi\} \quad \text{and} \quad E = \{(0, 0)\}.$$

Then

$$S^* \setminus E^* = \{(r, \theta) \in \mathbb{R}^2 : 0 < r \leq 1 \text{ and } 0 \leq \theta < 2\pi\} \quad \text{and} \quad S \setminus E = \{(x, y) \in \mathbb{R}^2 : 0 < x^2 + y^2 \leq 1\}.$$

We have shown in Example 6.2.8 that the function

$$T : S^* \setminus E^* \rightarrow S \setminus E : (r, \theta) \mapsto (r \cos \theta, r \sin \theta)$$

is one-to-one and onto.

**THEOREM 6A.** Suppose that  $S^* \subseteq \mathbb{R}^2$  and  $S \subseteq \mathbb{R}^2$  are elementary regions, and that the function  $T : S^* \rightarrow S$ , where  $T(u, v) = (x(u, v), y(u, v))$  for every  $(u, v) \in S^*$ , has continuous partial derivatives. Suppose further that  $T : S^* \rightarrow S$  is essentially one-to-one and onto. Then for any Riemann integrable function  $f : S \rightarrow \mathbb{R}$ , we have

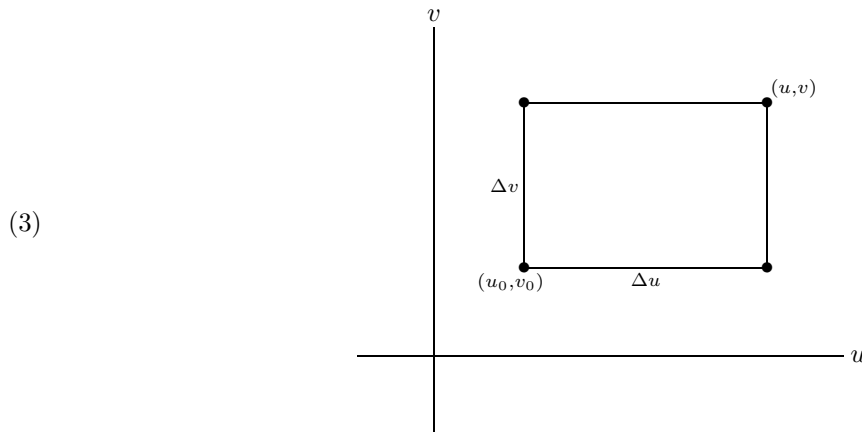
$$\iint_S f(x, y) dx dy = \iint_{S^*} f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv,$$

where the Jacobian determinant of  $T$  is given by

$$\frac{\partial(x, y)}{\partial(u, v)} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}.$$

REMARK. An alternative approach is to restrict  $S^*$  to open sets and take  $S = T(S^*)$ . However, the problem with this approach is that since  $S$  is usually given, it may not always be possible to find a suitable open set  $S^*$  such that  $T(S^*) = S$  precisely, and we face the same difficulty, albeit in a slightly different disguise.

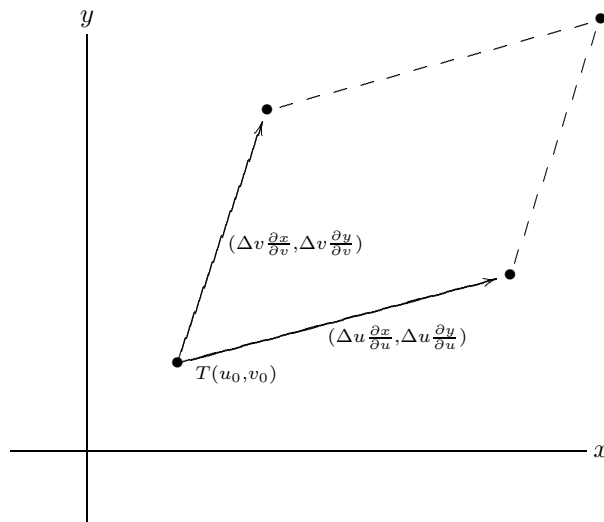
We shall not prove Theorem 6A. Instead we shall only briefly indicate the ideas. Consider a small rectangle



in  $S^*$  with bottom left vertex  $(u_0, v_0)$  and top right vertex  $(u, v) = (u_0 + \Delta u, v_0 + \Delta v)$ . We know from our discussion of differentiability in Chapter 2 that a good approximation to  $T(u, v)$  is given by the linear mapping

$$(4) \quad T(u_0, v_0) + (\mathbf{D}T)(u_0, v_0) \begin{pmatrix} u - u_0 \\ v - v_0 \end{pmatrix} = T(u_0, v_0) + (\mathbf{D}T)(u_0, v_0) \begin{pmatrix} \Delta u \\ \Delta v \end{pmatrix},$$

where  $(\mathbf{D}T)(u_0, v_0)$  denotes the total derivative of  $T$  at  $(u_0, v_0)$ . The image of the rectangle (3) under the mapping (4) is a parallelogram



with one vertex at  $T(u_0, v_0)$  and adjacent sides, corresponding to  $\Delta u$  and  $\Delta v$ , given by the vectors

$$(\mathbf{D}T)(u_0, v_0) \begin{pmatrix} \Delta u \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \begin{pmatrix} \Delta u \\ 0 \end{pmatrix} = \Delta u \begin{pmatrix} \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} \end{pmatrix}$$

and

$$(\mathbf{D}T)(u_0, v_0) \begin{pmatrix} 0 \\ \Delta v \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \begin{pmatrix} 0 \\ \Delta v \end{pmatrix} = \Delta v \begin{pmatrix} \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial v} \end{pmatrix}$$

respectively (here the partial derivatives are supposed to be evaluated at  $(u_0, v_0)$ ). Clearly the area of this parallelogram is

$$\left| \det \begin{pmatrix} \Delta u \frac{\partial x}{\partial u} & \Delta v \frac{\partial x}{\partial v} \\ \Delta u \frac{\partial y}{\partial u} & \Delta v \frac{\partial y}{\partial v} \end{pmatrix} \right| = \left| \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \right| \Delta u \Delta v = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \Delta u \Delta v.$$

EXAMPLE 6.3.3. Suppose that  $0 \leq \theta_1 < \theta_2 \leq 2\pi$ . Consider the region

$$S = \{(r \cos \theta, r \sin \theta) : \theta_1 \leq \theta \leq \theta_2 \text{ and } g_1(\theta) \leq r \leq g_2(\theta)\},$$

where  $g_1(\theta) \leq g_2(\theta)$  for every  $\theta \in [0, 2\pi]$ . Writing  $(x, y) = (r \cos \theta, r \sin \theta)$ , we can calculate the area of  $S$  as

$$A(S) = \iint_S dx dy = \iint_{S^*} \left| \frac{\partial(x, y)}{\partial(r, \theta)} \right| dr d\theta,$$

where

$$S^* = \{(r, \theta) \in \mathbb{R}^2 : \theta_1 \leq \theta \leq \theta_2 \text{ and } g_1(\theta) \leq r \leq g_2(\theta)\}.$$

Then

$$A(S) = \int_{\theta_1}^{\theta_2} \left( \int_{g_1(\theta)}^{g_2(\theta)} \left| \frac{\partial(x, y)}{\partial(r, \theta)} \right| dr \right) d\theta = \int_{\theta_1}^{\theta_2} \left( \int_{g_1(\theta)}^{g_2(\theta)} r dr \right) d\theta = \int_{\theta_1}^{\theta_2} \frac{g_2^2(\theta) - g_1^2(\theta)}{2} d\theta.$$

EXAMPLE 6.3.4. Consider the region

$$S = \{(x, y) \in \mathbb{R}^2 : x, y \geq 0 \text{ and } 1 \leq x^2 + y^2 \leq 4\}$$

in the first quadrant between the two circles of radii 1 and 2 centred at  $(0, 0)$ . We showed in Example 5.4.4 that

$$\iint_S (x^2 + y^2) dx dy = \frac{15\pi}{8}.$$

We now use the substitution  $x = r \cos \theta$  and  $y = r \sin \theta$  instead, and consider the rectangle

$$S^* = \{(r, \theta) \in \mathbb{R}^2 : 1 \leq r \leq 2 \text{ and } 0 \leq \theta \leq \pi/2\}.$$

Note that the function

$$T : S^* \rightarrow S : (r, \theta) \mapsto (r \cos \theta, r \sin \theta)$$

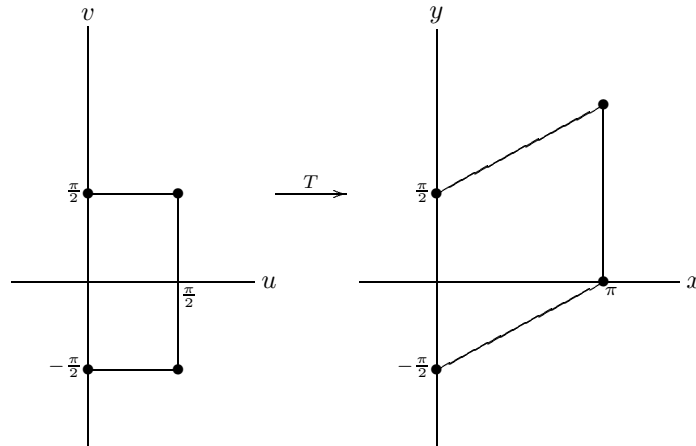
is one-to-one and onto. By Theorem 6A, we have

$$\iint_S (x^2 + y^2) \, dx \, dy = \iint_{S^*} (r^2 \cos^2 \theta + r^2 \sin^2 \theta) \left| \frac{\partial(x, y)}{\partial(r, \theta)} \right| \, dr \, d\theta = \iint_{S^*} r^3 \, dr \, d\theta = \dots = \frac{15\pi}{8}.$$

EXAMPLE 6.3.5. Consider the parallelogram  $S$  with vertices  $(\pi, 0)$ ,  $(\pi, \pi)$  and  $(0, \pm\pi/2)$ . We showed in Example 5.4.2 that

$$\iint_S \sin x \cos y \, dx \, dy = \frac{8}{3}.$$

We now use the substitution  $x = 2u$  and  $y = u + v$  instead, and consider the rectangle  $S^*$  with vertices  $(0, \pm\pi/2)$  and  $(\pi/2, \pm\pi/2)$ .



Note that the function

$$T : S^* \rightarrow S : (u, v) \mapsto (2u, u + v)$$

is one-to-one and onto. By Theorem 6A, we have

$$\begin{aligned} \iint_S \sin x \cos y \, dx \, dy &= \iint_{S^*} \sin 2u \cos(u + v) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \, du \, dv = 2 \iint_{S^*} \sin 2u (\cos u \cos v - \sin u \sin v) \, du \, dv \\ &= 2 \left( \int_0^{\pi/2} \sin 2u \cos u \, du \right) \left( \int_{-\pi/2}^{\pi/2} \cos v \, dv \right) - 2 \left( \int_0^{\pi/2} \sin 2u \sin u \, du \right) \left( \int_{-\pi/2}^{\pi/2} \sin v \, dv \right) \\ &= 2 \left( \int_0^{\pi/2} \sin 2u \cos u \, du \right) \left( \int_{-\pi/2}^{\pi/2} \cos v \, dv \right) = \dots = \frac{8}{3}. \end{aligned}$$

EXAMPLE 6.3.6. Consider the repeated integral

$$\int_0^1 \left( \int_0^1 (x^2 + y^2)^{3/2} \, dx \right) \, dy = \iint_R (x^2 + y^2)^{3/2} \, dx \, dy,$$

where  $R = [0, 1] \times [0, 1]$ . By splitting  $R$  into two triangles along the line  $y = x$ , it is easy to see that

$$\begin{aligned} \iint_R (x^2 + y^2)^{3/2} dx dy &= \int_0^1 \left( \int_0^x (x^2 + y^2)^{3/2} dy \right) dx + \int_0^1 \left( \int_0^y (x^2 + y^2)^{3/2} dx \right) dy \\ &= 2 \iint_S (x^2 + y^2)^{3/2} dx dy, \end{aligned}$$

where  $S$  is the triangular region with vertices  $(0, 0)$ ,  $(1, 0)$  and  $(1, 1)$ . We use the substitution  $x = r \cos \theta$  and  $y = r \sin \theta$ , and consider the region

$$S^* = \{(r, \theta) \in \mathbb{R}^2 : 0 \leq \theta \leq \pi/4 \text{ and } 0 \leq r \leq \sec \theta\}.$$

Note that the function

$$T : S^* \rightarrow S : (r, \theta) \mapsto (r \cos \theta, r \sin \theta)$$

is one-to-one and onto, provided that we remove the line  $r = 0$  from  $S^*$  and the point  $(x, y) = (0, 0)$  from  $S$ . By Theorem 6A, we have

$$\begin{aligned} \iint_S (x^2 + y^2)^{3/2} dx dy &= \iint_{S^*} (r^2 \cos^2 \theta + r^2 \sin^2 \theta)^{3/2} \left| \frac{\partial(x, y)}{\partial(r, \theta)} \right| dr d\theta = \int_0^{\pi/4} \left( \int_0^{\sec \theta} r^4 dr \right) d\theta \\ &= \frac{1}{5} \int_0^{\pi/4} \sec^5 \theta d\theta = \left[ \frac{1}{20} \tan \theta + \sec^3 \theta + \frac{3}{40} \tan \theta \sec \theta + \frac{3}{40} \log |\tan \theta + \sec \theta| \right]_0^{\pi/4} \\ &= \frac{7\sqrt{2}}{40} + \frac{3}{40} \log(1 + \sqrt{2}). \end{aligned}$$

#### 6.4. Triple Integrals

Our discussion so far can be extended to triple integrals with no extra complication.

**DEFINITION.** Suppose that  $S \subseteq \mathbb{R}^3$ . Then we say that a subset  $E \subseteq S$  is exceptional if  $E$  is contained in a finite union of surfaces in  $S$  representing one of the three variables as a continuous function of the other two variables.

**DEFINITION.** Suppose that  $S^*$  and  $S$  are sets in  $\mathbb{R}^3$ . Then we say that a function  $T : S^* \rightarrow S$  is essentially one-to-one and onto if there exist exceptional subsets  $E^* \subseteq S^*$  and  $E \subseteq S$  such that the function  $T : S^* \setminus E^* \rightarrow S \setminus E$  is one-to-one and onto.

**THEOREM 6B.** Suppose that  $S^* \subseteq \mathbb{R}^3$  and  $S \subseteq \mathbb{R}^3$  are elementary regions, and that the function  $T : S^* \rightarrow S$ , where  $T(u, v, w) = (x(u, v, w), y(u, v, w), z(u, v, w))$  for every  $(u, v, w) \in S^*$ , has continuous partial derivatives. Suppose further that  $T : S^* \rightarrow S$  is essentially one-to-one and onto. Then for any Riemann integrable function  $f : S \rightarrow \mathbb{R}$ , we have

$$\iiint_S f(x, y, z) dx dy dz = \iiint_{S^*} f(x(u, v, w), y(u, v, w), z(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du dv dw,$$

where the Jacobian determinant of  $T$  is given by

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix}.$$



REMARKS. (1) The image under  $T$  of a rectangular box in  $S^*$  with side lengths  $\Delta u$ ,  $\Delta v$  and  $\Delta w$  can be approximated by a parallelepiped in  $S$  with volume

$$\left| \det \begin{pmatrix} \Delta u \frac{\partial x}{\partial u} & \Delta v \frac{\partial x}{\partial v} & \Delta w \frac{\partial x}{\partial w} \\ \Delta u \frac{\partial y}{\partial u} & \Delta v \frac{\partial y}{\partial v} & \Delta w \frac{\partial y}{\partial w} \\ \Delta u \frac{\partial z}{\partial u} & \Delta v \frac{\partial z}{\partial v} & \Delta w \frac{\partial z}{\partial w} \end{pmatrix} \right| = \left| \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix} \right| \Delta u \Delta v \Delta w = \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| \Delta u \Delta v \Delta w.$$

(2) Cylindrical coordinates  $(r, \theta, z)$  are related to rectangular coordinates  $(x, y, z)$  in  $\mathbb{R}^3$  by

$$x = r \cos \theta, \quad y = r \sin \theta, \quad z = z.$$

Hence

$$\frac{\partial(x, y, z)}{\partial(r, \theta, z)} = \det \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{pmatrix} = \det \begin{pmatrix} \cos \theta & -r \sin \theta & 0 \\ \sin \theta & r \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = r,$$

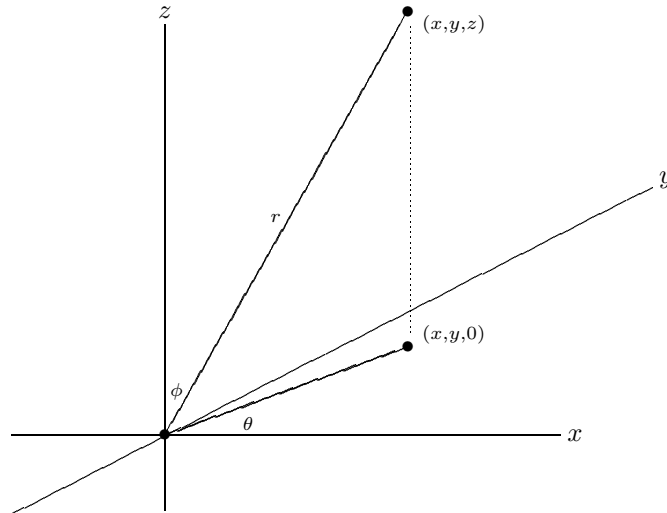
so that

$$\iiint_S f(x, y, z) \, dx \, dy \, dz = \iiint_{S^*} f(r \cos \theta, r \sin \theta, z) r \, dr \, d\theta \, dz.$$

Here, we essentially leave the variable  $z$  alone, and replace the variables  $x$  and  $y$  by the variables  $r$  and  $\theta$  in the same way as with rectangular and polar coordinates in  $\mathbb{R}^2$ .

(3) Spherical coordinates  $(r, \phi, \theta)$  are related to rectangular coordinates  $(x, y, z)$  in  $\mathbb{R}^3$  by

$$x = r \sin \phi \cos \theta, \quad y = r \sin \phi \sin \theta, \quad z = r \cos \phi.$$



We have

$$\begin{aligned}
\frac{\partial(x, y, z)}{\partial(r, \phi, \theta)} &= \det \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{pmatrix} = \det \begin{pmatrix} \sin \phi \cos \theta & r \cos \phi \cos \theta & -r \sin \phi \sin \theta \\ \sin \phi \sin \theta & r \cos \phi \sin \theta & r \sin \phi \cos \theta \\ \cos \phi & -r \sin \phi & 0 \end{pmatrix} \\
&= (\cos \phi) \det \begin{pmatrix} r \cos \phi \cos \theta & -r \sin \phi \sin \theta \\ r \cos \phi \sin \theta & r \sin \phi \cos \theta \end{pmatrix} + (r \sin \phi) \det \begin{pmatrix} \sin \phi \cos \theta & -r \sin \phi \sin \theta \\ \sin \phi \sin \theta & r \sin \phi \cos \theta \end{pmatrix} \\
&= (\cos \phi)(r^2 \cos \phi \sin \phi \cos^2 \theta + r^2 \cos \phi \sin \phi \sin^2 \theta) + (r \sin \phi)(r \sin^2 \phi \cos^2 \theta + r \sin^2 \phi \sin^2 \theta) \\
&= r^2 \sin \phi \cos^2 \phi + r^2 \sin \phi \sin^2 \phi = r^2 \sin \phi,
\end{aligned}$$

so that

$$\iiint_S f(x, y, z) \, dx dy dz = \iiint_{S^*} f(r \sin \phi \cos \theta, r \sin \phi \sin \theta, r \cos \phi) r^2 \sin \phi \, dr d\phi d\theta.$$

EXAMPLE 6.4.1. Consider the integral

$$\iiint_S (x^2 + y^2 + z^2)^{1/2} \, dx dy dz,$$

where

$$S = \{(x, y, z) \in \mathbb{R}^3 : y, z \geq 0 \text{ and } 4 \leq x^2 + y^2 + z^2 \leq 9\}.$$

We use spherical coordinates and the substitution

$$x = r \sin \phi \cos \theta, \quad y = r \sin \phi \sin \theta, \quad z = r \cos \phi,$$

and consider the set

$$S^* = \left\{ (r, \phi, \theta) \in \mathbb{R}^3 : r \in [2, 3], \phi \in \left[0, \frac{\pi}{2}\right] \text{ and } \theta \in [0, \pi] \right\}.$$

Note that the function

$$T : S^* \rightarrow S : (r, \phi, \theta) \mapsto (r \sin \phi \cos \theta, r \sin \phi \sin \theta, r \cos \phi)$$

is one-to-one and onto. By Theorem 6B and noting Remark (3) above, we have

$$\begin{aligned}
\iiint_S (x^2 + y^2 + z^2)^{1/2} \, dx dy dz &= \iiint_{S^*} (r^2 \sin^2 \phi \cos^2 \theta + r^2 \sin^2 \phi \sin^2 \theta + r^2 \cos^2 \phi)^{1/2} r^2 \sin \phi \, dr d\phi d\theta \\
&= \iiint_{S^*} r^3 \sin \phi \, dr d\phi d\theta = \left( \int_2^3 r^3 \, dr \right) \left( \int_0^{\pi/2} \sin \phi \, d\phi \right) \left( \int_0^\pi d\theta \right) = \frac{65\pi}{4}.
\end{aligned}$$

EXAMPLE 6.4.2. Consider the integral

$$\iiint_S (x^2 + y^2 + z^2)^{5/2} dx dy dz,$$

where

$$S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}.$$

We use spherical coordinates and the substitution

$$x = r \sin \phi \cos \theta, \quad y = r \sin \phi \sin \theta, \quad z = r \cos \phi,$$

and consider the set

$$S^* = \{(r, \phi, \theta) \in \mathbb{R}^3 : r \in [0, 1], \phi \in [0, \pi] \text{ and } \theta \in [0, 2\pi]\}.$$

Note that the function

$$T : S^* \rightarrow S : (r, \phi, \theta) \mapsto (r \sin \phi \cos \theta, r \sin \phi \sin \theta, r \cos \phi)$$

is one-to-one and onto, provided that we remove the planes  $r = 0$  and  $\theta = 2\pi$  from  $S^*$  and the point  $(0, 0, 0)$  from  $S$ . By Theorem 6B and noting Remark (3) above, we have

$$\iiint_S (x^2 + y^2 + z^2)^{5/2} dx dy dz = \iiint_{S^*} r^7 \sin \phi dr d\phi d\theta = \left( \int_0^1 r^7 dr \right) \left( \int_0^\pi \sin \phi d\phi \right) \left( \int_0^{2\pi} d\theta \right) = \frac{\pi}{2}.$$

EXAMPLE 6.4.3. Consider the integral

$$\iiint_S (x^2 + y^2)^{1/2} z dx dy dz,$$

where

$$S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq 1 \text{ and } 0 \leq z \leq 1\}.$$

We use cylindrical coordinates and the substitution

$$x = r \cos \theta, \quad y = r \sin \theta, \quad z = z,$$

and consider the set

$$S^* = \{(r, \theta, z) \in \mathbb{R}^3 : r \in [0, 1], \theta \in [0, 2\pi] \text{ and } z \in [0, 1]\}.$$

Note that the function

$$T : S^* \rightarrow S : (r, \theta, z) \mapsto (r \cos \theta, r \sin \theta, z)$$

is one-to-one and onto, provided that we remove the planes  $r = 0$  and  $\theta = 2\pi$  from  $S^*$  and the point  $(0, 0, 0)$  from  $S$ . By Theorem 6B and noting Remark (2) above, we have

$$\begin{aligned} \iiint_S (x^2 + y^2)^{1/2} z dx dy dz &= \iiint_{S^*} (r^2 \cos^2 \theta + r^2 \sin^2 \theta)^{1/2} z r dr d\theta dz \\ &= \left( \int_0^1 r^2 dr \right) \left( \int_0^{2\pi} d\theta \right) \left( \int_0^1 z dz \right) = \frac{\pi}{3}. \end{aligned}$$

## PROBLEMS FOR CHAPTER 6

1. By changing to polar coordinates, evaluate the integral  $\int_0^1 \left( \int_0^{\sqrt{1-x^2}} e^{-x^2-y^2} dy \right) dx$ .
2. By writing  $2x = r - s$  and  $2y = r + s$ , evaluate the integral  $\int_0^{1/2} \left( \int_x^{1-x} \left( \frac{x-y}{x+y} \right)^2 dy \right) dx$ .
3. By using the transformation  $x + y = u$  and  $y = uv$ , show that  $\int_0^1 \left( \int_0^{1-x} e^{y/(x+y)} dy \right) dx = \frac{e-1}{2}$ .  
[REMARK: Take great care when calculating the Jacobian.]

4. Suppose that  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  and  $g : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  are differentiable. Suppose further that  $(u, v) = f(s, t)$  and  $(x, y) = g(u, v)$ , so that  $(x, y) = (g \circ f)(s, t)$ .  
a) Use the Chain rule  $(\mathbf{D}(g \circ f))(s, t) = (\mathbf{D}g)(u, v)(\mathbf{D}f)(s, t)$  to show that

$$\frac{\partial(x, y)}{\partial(s, t)} = \frac{\partial(x, y)}{\partial(u, v)} \frac{\partial(u, v)}{\partial(s, t)}.$$

- b) Show that  $\frac{\partial(x, y)}{\partial(u, v)} \frac{\partial(u, v)}{\partial(x, y)} = 1$  provided that  $\frac{\partial(x, y)}{\partial(u, v)} \neq 0$ .
5. Let  $S$  be the region in the first quadrant of  $\mathbb{R}^2$  bounded by  $x^2 - y^2 = 1$ ,  $x^2 - y^2 = 9$ ,  $xy = 2$  and  $xy = 4$ . Consider also the substitution  $u = x^2 - y^2$  and  $v = 2xy$ .  
a) Draw a picture of the region  $S$ .  
b) Use the result in Question 4(b) to find  $\partial(x, y)/\partial(u, v)$ .  
c) Show that  $2x^2 = (u^2 + v^2)^{1/2} + u$  and  $2y^2 = (u^2 + v^2)^{1/2} - u$ .  
d) Let  $S^* = \{(u, v) \in \mathbb{R}^2 : u \in [1, 9] \text{ and } v \in [4, 8]\}$ . Show that  $T : S^* \rightarrow S : (u, v) \mapsto (x, y)$  is one-to-one and onto.  
e) By changing variables, show that  $\iint_S (x^2 + y^2) dx dy = 8$ .

6. Evaluate each of the following triple integrals by using a suitable substitution:

- a)  $\iiint_S (x^2 + y^2 + z^2) dx dy dz$ , where  $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq 4 \text{ and } 0 \leq z \leq 2\}$
- b)  $\iiint_S (x^2 + y^2 + z^2)^{1/2} e^{-(x^2+y^2+z^2)} dx dy dz$ , where  $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 4\}$
- c)  $\iiint_S \frac{dx dy dz}{(x^2 + y^2 + z^2)^{3/2}}$ , where  $S = \{(x, y, z) \in \mathbb{R}^3 : a^2 \leq x^2 + y^2 + z^2 \leq b^2\}$  with  $b > a > 0$

7. Let  $S$  be the region in  $\mathbb{R}^3$  bounded by the paraboloid  $z = x^2 + y^2$  and the cone  $z = \sqrt{x^2 + y^2}$ .  
a) Find the set of points  $(x, y, z) \in \mathbb{R}^3$  where the paraboloid intersects the cone.  
b) Draw the cross section of the region  $S$  on the plane  $y = 0$ .  
c) Show that the volume of  $S$  is equal to

$$\iint_T (\sqrt{x^2 + y^2} - (x^2 + y^2)) dx dy,$$

where  $T = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ .

- d) By changing to polar coordinates, show that the volume of  $S$  is equal to  $\pi/6$ .

8. By considering a suitable set of coordinates in  $\mathbb{R}^3$  and making the necessary change of variables, show that

$$\iiint_S \sqrt{x^2 + y^2 + z^2} \, dx \, dy \, dz = \frac{27\pi(2\sqrt{2} - 1)}{2},$$

where  $S$  is the region in  $\mathbb{R}^3$  bounded by the plane  $z = 3$  and the cone  $z = \sqrt{x^2 + y^2}$ .